Time ⇔ Frequency Transformations

1 Introduction

Usually it is desirable that radio communications systems operate within a prescribed band of frequencies so as not to cause interference with other systems using the available spectrum. In order to understand radio communication systems then, it is important to be able to calculate the frequencies associated with a particular signal. In many cases the time behaviour of the signal is known (or can be measured with a CRO, for example) but how can we find the spectral description of the signal from this representation? The answer of course is to use a time to frequency transformation. These transformations should not be new to you and the following is meant only as a brief review of this topic to assist the course being taught here.

2 Fourier Series

If a waveform is periodic (period T) it can be represented by a sum of sinusoidal terms.

Trigometric Form

Let f(t) be a periodic time signal then:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \quad \text{Eqn. 1}$$

where $\omega_0 = \frac{2\pi}{T}$ and **a** and **b** are **real constants** such that:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t \;dt \quad \text{and} \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t \;dt \qquad \text{Eqn. 2}$$

and $$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt$$ The DC average of the signal is $a_0/2$. Eqn. 3

Exponential (or Complex) Form

$$f(t)\sum_{n=-\infty}^{\infty} = c_n\, e^{jn\omega_0 t} \qquad \text{Eqn. 4}$$

Note : The summation covers negative as well as positive values of n. $C_{\mathbf{n}}$ are **complex constants** such that:

$$C_0 = a_0$$

$$C_n = \frac{a_n - jb_n}{2}$$ when n is positive and Eqn. 5

$$C_{-n} = \frac{a_n + jb_n}{2}$$ when n is negative.

or $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\,dt$$ where n = 0, ±1, ±2, ±3 etc Eqn. 6

Note From these expressions, if we know ω_0 and the C_n values (or the corresponding a and b values), we have a complete knowledge of time waveform f(t). Either f(t) or C_n fully specifies the waveform. Sometimes it is easier to work in the time domain and at other times it is easier to work in the frequency domain.

Example

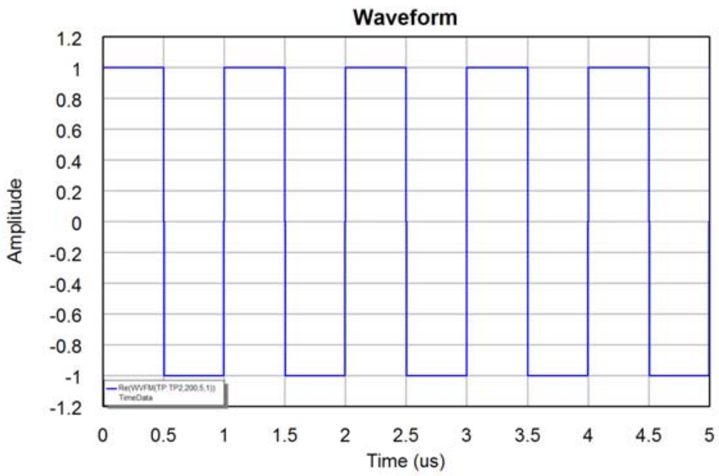

Figure 1. Square-wave Time domain waveform.

For example, consider a square wave time domain waveform, as shown in Figure 1. The frequency of the squarewave is 1 MHz, corresponding to a period of 1 μs. The corresponding frequency spectrum can be determined by evaluating C_n as follows:

f(t) = A if -T/4 > t > T/4 Eqn .7

f(t) = -A if T/4 > t > 3T/4

substituting this in equation 6 gives:

$$C_n = \frac{1}{T}\left[\int_{-T/4}^{T/4} Ae^{-jn\omega_0 t}dt + \int_{T/4}^{3T/4} -Ae^{-jn\omega_0 t}dt\right]$$

$$= \frac{1}{-jn\omega_0 T}\left[A(e^{jn\omega_0 T/4} - e^{-jn\omega_0 T/4}) - A(e^{jn\omega_0 3T/4} - e^{jn\omega_0 T/4})\right] \qquad \text{Eqn. 8}$$

$$= \frac{2A}{-jn\omega_0 T}\left[e^{-jn\omega_0 T/4} - e^{jn\omega_0 T/4}\right]$$

when n is odd.

Since $\mathrm{Sin}(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$ and $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$ or $\omega_0 T = 2\pi$ Eqn. 9

$$C_n = \frac{4A}{-n\pi}\left[\frac{e^{-jn\pi/2} - e^{jn\pi/2}}{2j}\right] = \frac{4A\sin(n\pi/2)}{n\pi}$$

$$= 2A\frac{Sin(n\pi/2)}{n\pi/2}$$

where -∞ > n=odd > ∞ Eqn. 10

So that: $C_n = \dfrac{4A}{n\pi}$ when n = ±(1, 5, 9 etc)

$$C_n = \frac{-4A}{n\pi} \qquad \text{when n} = \pm(3, 7, 11 \text{ etc}) \qquad \text{Eqn. 11}$$

$$C_n = 0 \qquad \text{when n is even.}$$

The third harmonic is thus one third of the fundamental component or -9.54 dB below the fundamental. Figure 2 shows the spectrum of the square-wave of figure 1. From the markers, it can be seen that the third harmonic is -9.54 dB below the fundamental, as expected.

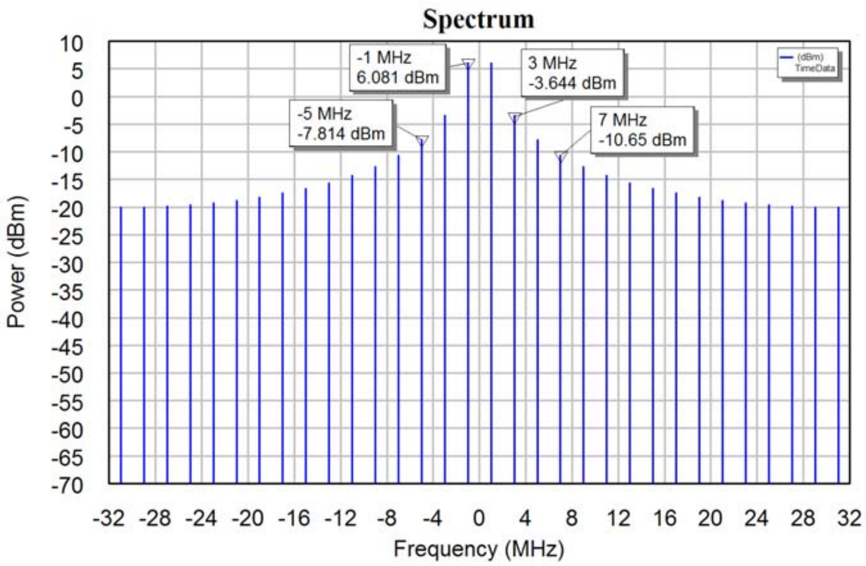

Figure 2. Spectrum of the Square Wave of figure 1.

The C_n components correspond to the frequency components of the spectrum shown in figure 2. Since the frequency of the square-wave is 1 MHz, the spacing of the spectral components is 1 MHz. C_0 will thus occur at 0 Hz (DC), C_1 will occur at 1 MHz, C_{-1} will occur at -1 MHz, C_2 will occur at 2 MHz, C_{-5} will occur at -5 MHz and so on.

A Cathode Ray Oscilloscope shows the time domain waveforms and a Spectrum Analyser shows the corresponding frequency spectrum. Since most spectrum analysers use simple filters to measure the amplitude (and thus power) of the different spectral components, the phase information is not always determined.

A square wave can thus be made up from a series of sinewaves as shown in figure 3.

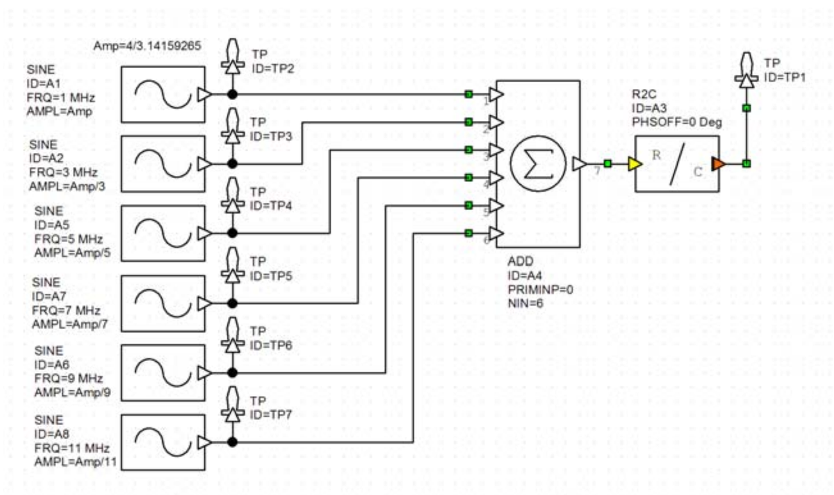

Figure 3. Constructing a square wave from sine-waves.

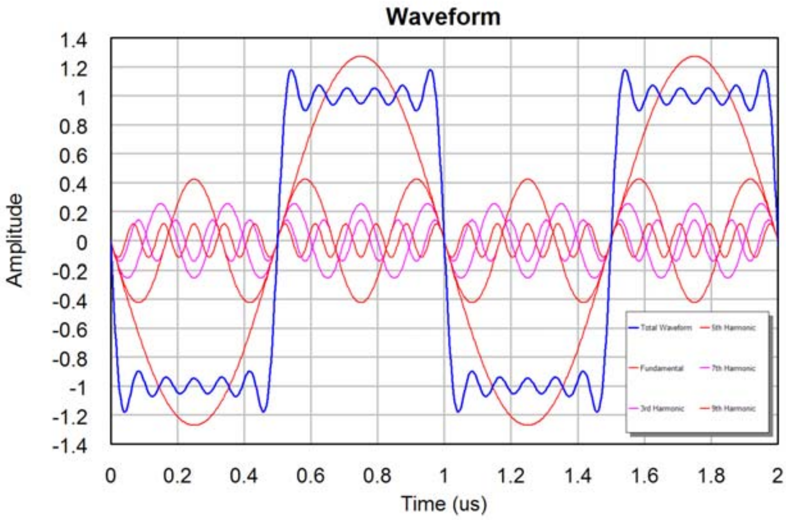

Figure 4. Waveforms obtained from the circuit of figure 3.

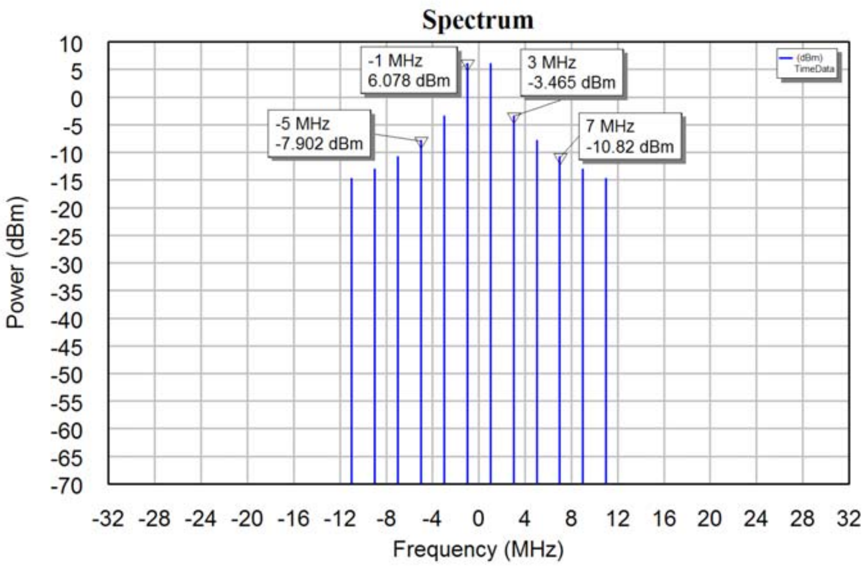

Figure 5. Spectrum for the waveform of figure 4.

Notes :

$|C_n| = |C_{-n}|$ so that the magnitudes of the positive and negative spectral components are the same, as can be seen in figure 2.

$C_n = C_{-n}^*$ where * indicates the complex conjugate, so that the phase of the positive and negative frequency components are of opposite phase.

The spectrum of figure 2 does not show the phase of the C_n values, only the amplitude, but often the magnitude is more important and this information is not needed. Because the positive and negative spectral components of figure 2 have the same amplitude and opposite phase, often only the positive frequency spectrum is shown.

Since $\mathrm{Sin}(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$, and the first term corresponds to a positive frequency and the second term corresponds to a negative frequency, any real waveform will always have both **positive** and **negative frequency** components. This is further illustrated by calculating the C_n values for a sinewave.

If $f(t) = A \ \mathrm{Cos}(\omega_o t + \phi)$

Then $C_n = 0$ when $|n| \neq 1$

$$C_1 = \frac{A}{2} e^{j(\phi)} = \frac{A}{2}\left[\cos(\phi) + j\sin(\phi)\right] \qquad \text{Eqn. 12}$$

$$C_{-1} = \frac{A}{2} e^{-j(\phi)} = \frac{A}{2}\left[\cos(-\phi) + j\sin(-\phi)\right] = \frac{A}{2}\left[\cos(\phi) - j\sin(\phi)\right] \qquad \text{Eqn. 13}$$

Thus C_{+1} and C_{-1} are complex conjugate as expected, but $f(t) = A\cos(\omega o\, t + \phi)$ is real.

Complex Time Waveforms

As we will see later in this course, sometimes a positive only frequency spectrum is required, where all the negative frequency components are zero. From equation 12, the positive frequency component is $\frac{A}{2}\left[\cos(\phi) + j\sin(\phi)\right]$, which corresponds to a complex time domain waveform. In addition the real and imaginary parts of this waveform with positive frequency components consists of two waveforms Cos(ϕ) and Sin(ϕ), which have a 90 degrees phase difference. Using Digital Signal Processing one can generate waveforms which satisfy this 90 degree phase difference. They are normally called *I* (in Phase) and *Q* (quadrature) components of the resulting complex time waveform.

Many of the modern communication techniques use I and Q modulation, to have independent control of the positive and negative spectral frequency components.

Examples

The time waveform in figure 6 is the same waveform as shown in figure 1, except that the transitions from the positive to the negative voltage levels are made more gradually. Figure 7 shows the corresponding spectrum. It can be seen that by making the transition one fifth of the period (i.e. 0.2 μs) then the fifth harmonic component becomes very small. In addition it can be seen that the higher frequency components, say the 20 MHz to 30 MHz frequency range, are much smaller in figure 7.

More gradual transitions will thus reduce the bandwidth required to contain most of the energy in the waveform. This principle applies for all waveforms, and the principle of shaping the time waveform to reduce the bandwidth occupied is used for most communication systems in this course.

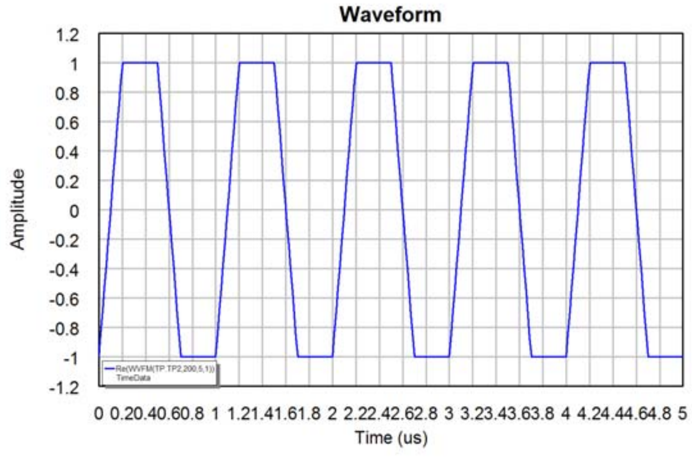

Figure 6. Periodic waveform with 20% rise and fall times.

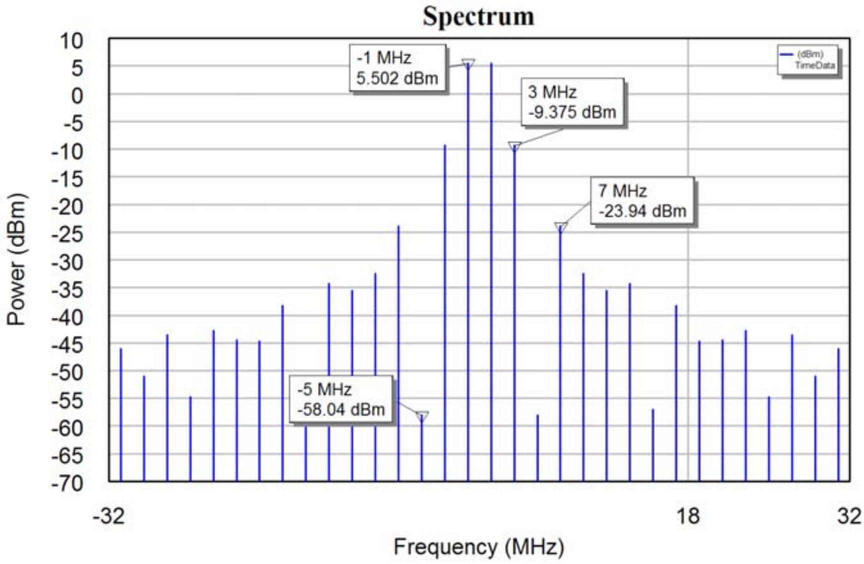

Figure 7. Spectrum of the waveform shown in figure 3.

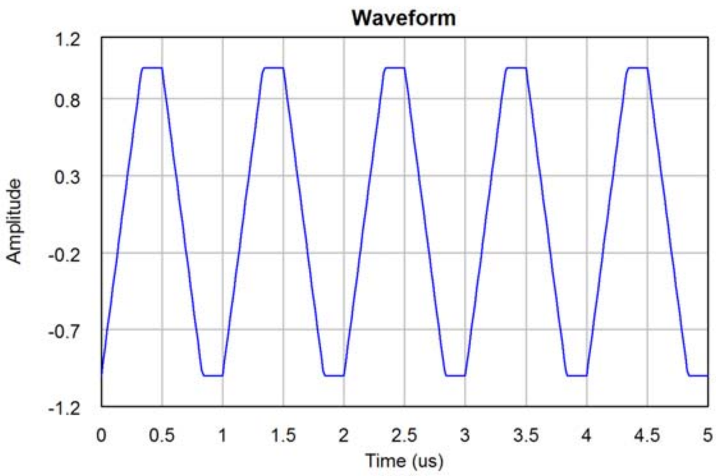

Figure 8. Periodic waveform with 33.33% rise and fall times.

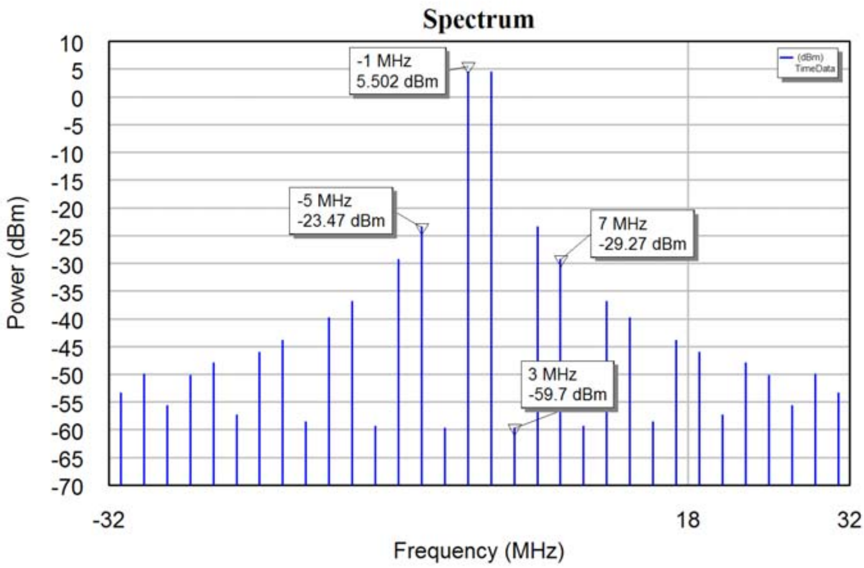

Figure 9. Spectrum of the waveform shown in figure 5.

Figure 8, shows a time waveform with a transition time of one third of the period. The corresponding spectrum shown in figure 9 shows this waveform has a very small third harmonic component. The fifth harmonic is -27.9 dB, i.e. 0.16% of the power of the fundamental. More than 99% of the power in the waveform is thus at the fundamental frequency, so that this waveform is a good approximation to a sinewave.

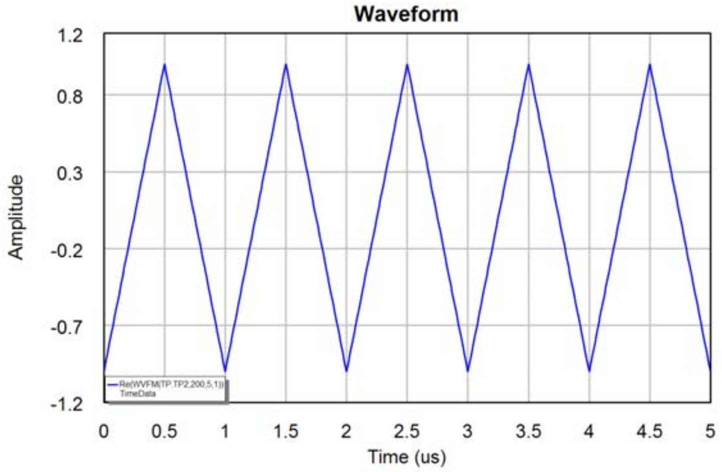

Figure 10. Triangular Periodic waveform.

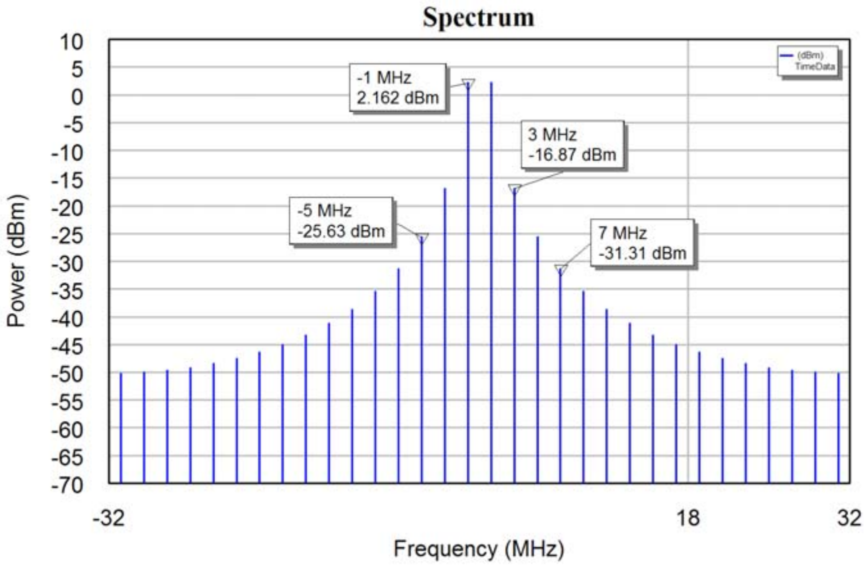

Figure 11. Spectrum of the waveform shown in figure 7.

The triangular waveform shown in figure 10 has a transition time of half the period. This results in the spectrum shown in figure 11. By calculating the resulting C_n values, it can be shown that the harmonics decrease proportional to n^2, so that the third harmonic is one ninth of the fundamental amplitude, corresponding to a 19.1 dB difference between the fundamental and the third harmonic, as can also be verified from figure 11. The triangular wave will thus have significantly smaller higher harmonics than a square wave, as can be seen by comparing figure 2 and figure 11.

3 Line Spectra and Average Power

In section 2, the relationship between the time and frequency domains have been demonstrated for a periodic waveform. We can find the power (rate of transfer of energy) associated with a time function but what is the relationship between the spectrum and the power contained in the waveform?

Let the voltage V be a periodic function of time such that, V = x(t). Then define the average power associated with this voltage to be the power that would be delivered to a 1Ω load.

Therefore, Power = Average $[x^2(t)]$

$$\text{Average}\left[\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t} \times \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega t}\right] \qquad \text{Eqn. 13}$$

Since the average of a cyclic function will be zero, we can see that

$$\text{Average}\left[\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t} \times \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega t}\right] = 0 \qquad \text{If } n \neq k,$$

And

$$\text{Average}\left[\sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t} \times \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega t}\right] = C_n\, C_k \qquad \text{if } n = -k. \qquad \text{Eqn. 14}$$

Thus since, $C_n = C_{-n}^{*}$

the **Power** $\mathbf{P} = \sum_{n=-\infty}^{\infty} c_n \,.\, c_{-n} = \sum_{n=-\infty}^{\infty} | c_n |^2$

The power contained in a signal is thus the sum of the spectral components squared.

The RMS voltage is $V_{rms} = \sqrt{\sum_{n=-\infty}^{\infty} | c_n |^2}$ Eqn. 15

This is one reason why one can be sure that the more than 99% of the total energy in the waveform of figure 5 is contained in the fundamental components, i.e. C_{+1} *and* C_{-1} .

Example

For a single sinusoid with a voltage of peak amplitude A [ie. X(t) = A cos($\omega_o\, t$)] , then

$C_1 = \frac{1}{2}A$ and $C_{-1} = \frac{1}{2}A$ while $C_n = 0$, when $n \neq \pm 1$.

$$\text{Hence the Power} \quad P = \sum_{n=-\infty}^{\infty} | C_n |^2 = |C_1|^2 + |C_{-1}|^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2} \qquad \text{Eqn. 16}$$

Note this is the same result as would be expected using the well known expression for the power produced by a ac voltage, that is $P = \frac{V_{rms}^2}{R}$ when R = 1 Ω.

4 The Fourier Transform

The Fourier series allows the spectrum of a periodic function to be calculated. However, the Fourier series is only applicable when we have a **periodic** function. How can we treat a **non-periodic** function like a single pulse or a random waveform?

There are two possible ways:

4.1 Window functions

The time waveform over a time interval from 0 to T is multiplied, or windowed with a time domain waveform which is zero at time 0 and time T and is a maximum at time T/2, as shown in Figure 12.

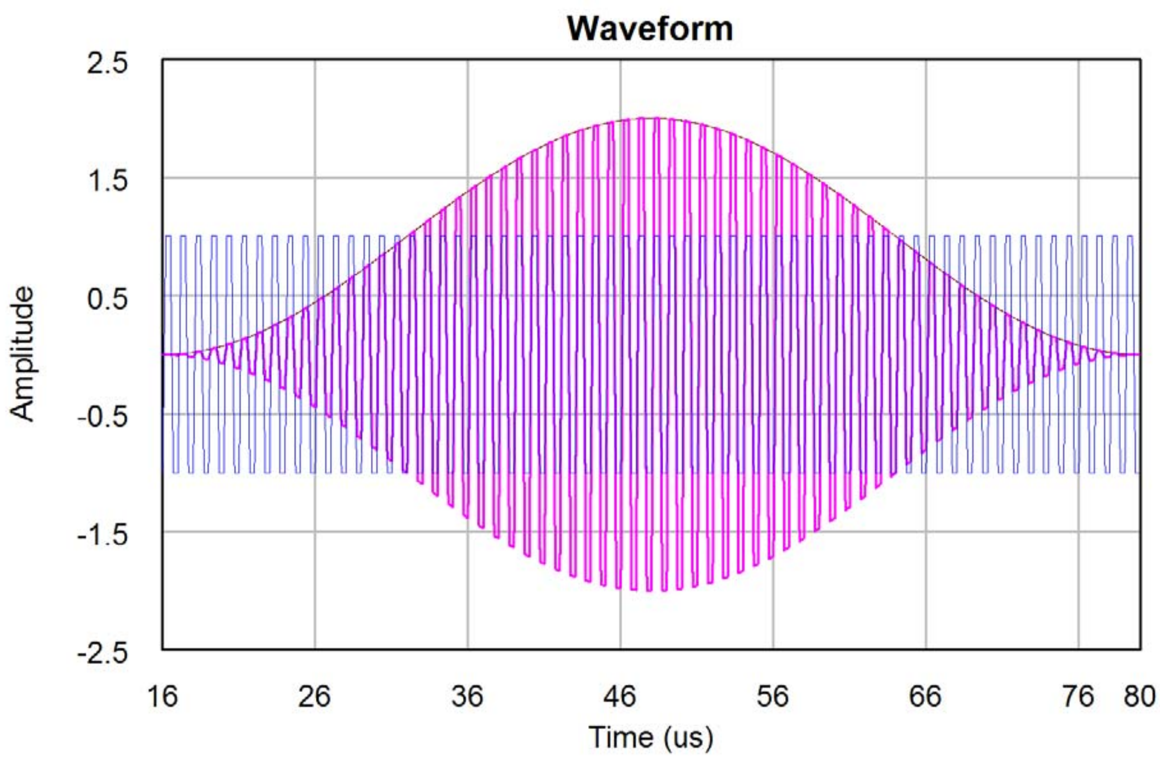

Figure 12. Window function applied to the time domain.

Window functions make the time function such that is can be repeated. The multiplication of the time waveform by the window function, spreads the energy to adjacent spectral lines, so that the resulting spectrum is not exactly that of the original spectrum. Window functions are outside the scope of EE3700 and are treated in detail in EE4000.

4.2 Fourier Transform.

If we allow the period T of a periodic function to tend towards infinitely, we use this infinitely long time period to accurately represent obtain a non-periodic function. By replacing T/2 in equation 6 with ∞ and scaling the integral by 1/T, one can define the Fourier Transform as:

$$g(\omega) = \int_{-\infty}^{\infty} \mathrm{f(t)}\ \mathrm{e}^{-j\omega t}\ \mathrm{dt} \quad \text{or} \quad g(f) = \int_{-\infty}^{\infty} \mathrm{f(t)}\ \mathrm{e}^{-j2\pi ft}\ \mathrm{dt} \qquad \text{Eqn. 17}$$

The Inverse Fourier Transform can similarly be obtained from equation 4 as:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\ e^{j\omega t}\ d\omega = \int_{-\infty}^{\infty} g(f)\ e^{j2\pi ft}\ df \qquad \text{Eqn. 18}$$

The frequency function g(ω) is called the **Spectral Density** and has units of volts/Hz.

5 Energy and the Spectral Density

Just as in the case of a periodic signal, we might try to define the **power** associated with a non-periodic signal. For example,

Power = Average [$f^2(t)$]

$$= \text{limit as } T \to \infty \ \frac{1}{T}\int_{0}^{T} f^2(t)\, dt \qquad \text{Eqn. 19}$$

But, note that since the time function, f(t), does not repeat and T → ∞, the Average Power will tend to zero!

Thus rather than talking about average power in this case, we are better to consider the finite **energy** contained in the signal. Thus define the energy produced by our pulse as;

$$Energy = \int_{-\infty}^{\infty} f^2(t)\ dt, \text{ which does not tend to zero.}$$

$$Energy = \int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} \frac{1}{2\pi} g(\omega)\ e^{j\omega t}\ d\omega\right] dt \qquad \text{Eqn. 20}$$

or rearranging,

$$Energy = \int_{-\infty}^{\infty} \frac{g(\omega)}{2\pi}\left[\int_{-\infty}^{\infty} f(t)\, e^{-(-j\omega t)}\ dt\right] d\omega$$

$$= \int_{-\infty}^{\infty} \frac{g(\omega)}{2\pi}\ g(-\omega)\ d\omega \qquad \text{Eqn. 21}$$

$$\text{Thus Energy} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g(\omega)|^2\ d\omega \qquad \text{Eqn. 22}$$

$$\text{or Energy} = \int_{-\infty}^{\infty} |g(f)|^2\ df \ \text{ - Rayleigh's Energy Formula.} \qquad \text{Eqn. 23}$$

You will therefore see that g(f) is a measure of how the energy contained in the pulse is distributed in the frequency domain.

6 Pulse Waveform

Repetitive Time Waveform

Firstly consider a waveform consisting of a pulse of amplitude A, pulse width τ and period T, as shown in figure 13. This is a repetitive waveform, so that a Fourier Series calculation is applied. The spectral components can thus be calculated using equation 6 as follows;

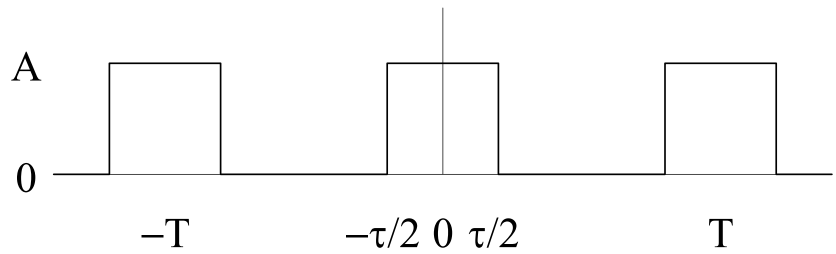

Figure 13. Time waveform for a periodic pulse

$$C_n = \frac{1}{T}\left[\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}dt\right] = \frac{A}{-jn\omega_0 T}\left[e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}\right]$$

$$= \frac{2A}{n\pi T} Sin(\frac{n\omega_0\tau}{2})$$ Eqn. 24

$$C_n = \frac{\tau A}{T}\frac{Sin(n\pi f_0\tau)}{(n\pi f_0\tau)}$$ Eqn. 25

Since $$Sinc(\lambda) = \frac{Sin(\pi\lambda)}{\pi\lambda}$$ Eqn. 26

$$C_n = \frac{\tau A}{T}\text{sinc}(nf_0\tau) = \frac{\tau A}{T}\text{sinc}(\frac{n\tau}{T})$$ Eqn. 27

The sine function has zeros when $\frac{n\pi\tau}{T} = m\pi$,where m= ±1, ±2, ±3, ±4 etc. If τ = T/5 then spectral nulls will occur at the 5th harmonic, the 10th harmonic etc. The spectrum has spectral lines corresponding to the fundamental and harmonic components. The resulting spectrum will thus look like the spectrum shown in figure 14.

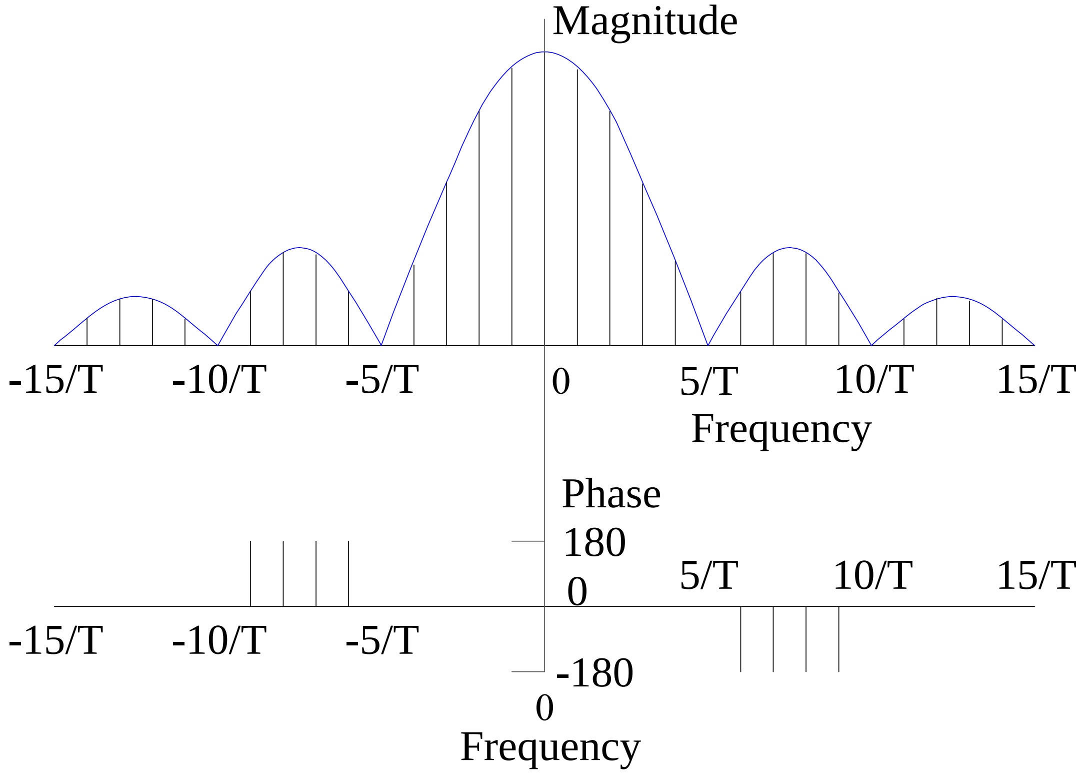

Figure 14. Magnitude and Phase of the spectral components C_n for the waveform of figure 13.

If the pulse width becomes narrower, the location of the nulls moves to higher frequencies. This means that narrower pulses occupy wider bandwidths.

Single Pulse in time

Now consider a single pulse of amplitude A and width τ. This pulse is not repetitive, so the Fourier Series cannot be applied, but the Fourier Transform can be used to calculate the power spectral density, to show the distribution of energy in the frequency domain. The power spectral density is calculated using equation 17 as:

$$g(\omega) = \int_{-\infty}^{\infty} f(t)\ e^{-j\omega t}\ dt = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} = A\tau \frac{Sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}} = A\tau\ \mathrm{sinc}(f\tau) \qquad \text{Eqn. 28}$$

Note that this is very similar to equation 27 when nf_0 is replaced with f. The resulting spectral density has the same shape as figure 14, but is now a continuous spectrum rather than a line spectrum, as shown in figure 15.

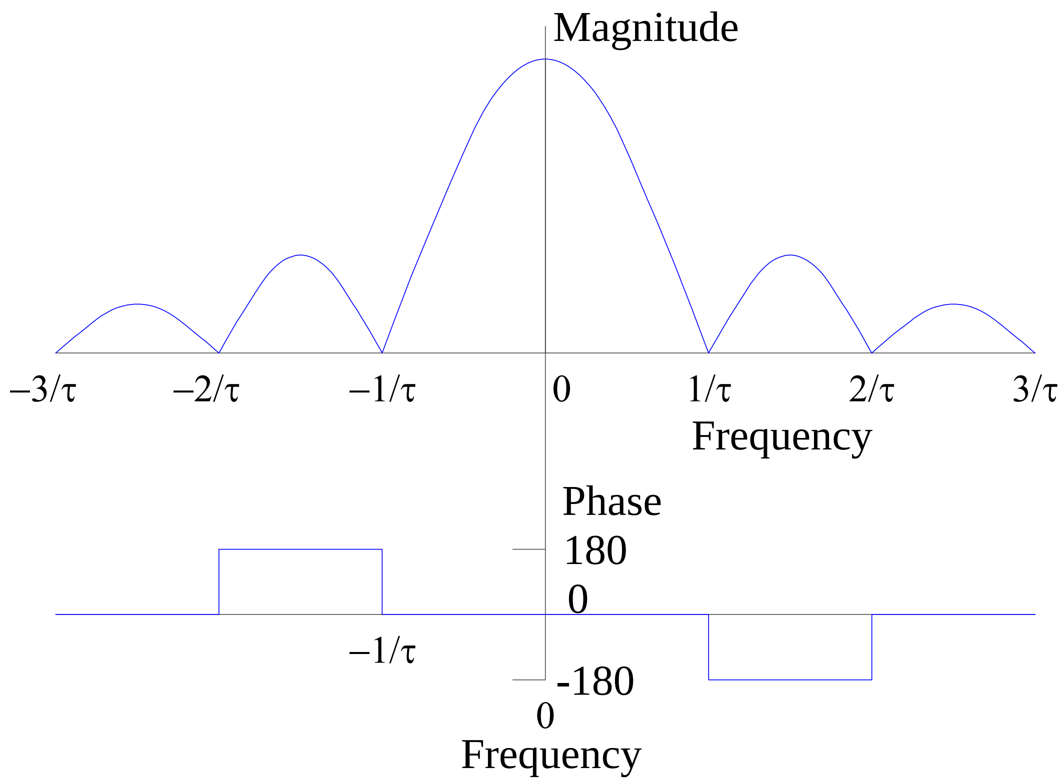

Figure 15. Magnitude and Phase of the power spectral density of a single pulse of width τ.

Ideal Filter

Now consider an ideal filter, as shown in figure 16. The phase of the filter is zero for all frequency. When a signal with a frequency below W is applied to the filter, it is passed without any phase shift. A signal with a frequency above W is stopped completely.

When a unit impulse is applied to the filter, the frequency response of the unit impulse is a constant, independent of frequency, so that the impulse response of the filter in the frequency domain is the same as figure 16. What does the impulse response look like in the time domain?

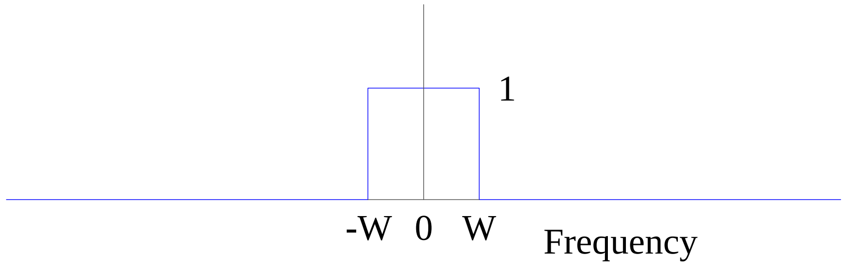

Figure 16. Ideal Filter frequency response.

The time waveform corresponding to the power spectral density of figure 16 is obtained using the inverse Fourier transform of equation 18. The impulse response of the filter is thus:

$$f(t) = \int_{-W}^{W} \mathrm{e}^{\,\mathrm{j}2\pi ft}\,\mathrm{d}f = \frac{1}{2j\pi t}\left[\mathrm{e}^{\,\mathrm{j}2\pi \mathrm{Wt}} - \mathrm{e}^{\,\mathrm{j}2\pi \mathrm{Wt}}\right] = 2W\,\frac{Sin(2\pi Wt)}{2\pi Wt} \qquad \text{Eqn. 29}$$

$$f(t) = 2W\ \mathrm{sinc}(2Wt)$$

The impulse response is thus a sinc function in time, as shown in figure 17.

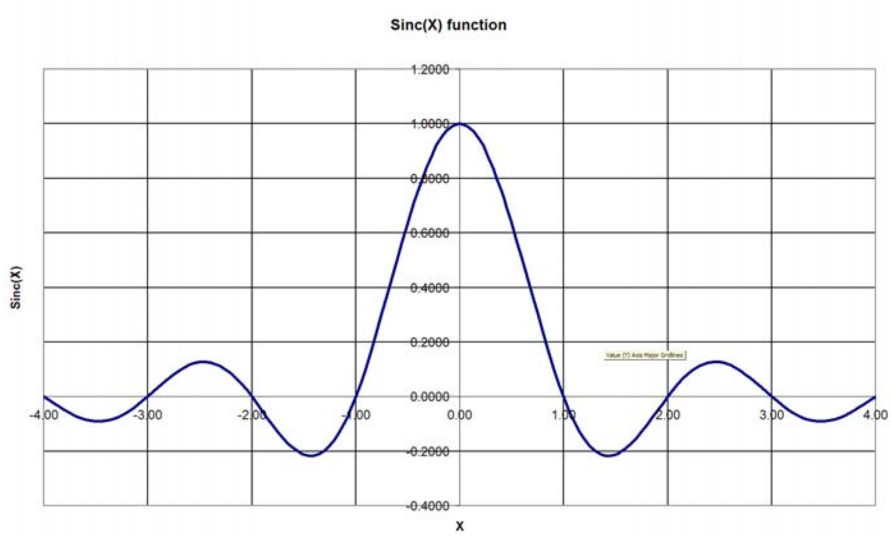

Figure 17 Sinc function. The X axis corresponds to time.

This example illustrates two important aspects:

Time and Frequency are dual domains. Equation 17 and 18 are the same with just time and frequency changed. A pulse in the time domain causes a sinc response in the frequency domain and vice versa.

For a zero phase filter, as is shown in figure 16, as shown in figure 17, the impulse response starts before time =0. The filter is thus clairvoyant and knows when an impulse is going to be applied. Clearly a practical filter must have h(t) = 0 for t < 0, since no filter can take action before the input signal exists. This means that we cannot construct filters to have an ideal amplitude response with no phase shift. In practice one can approximate an ideal filter, by applying a delay (time shift of figure 17), such that only very small amplitudes exist for time t < 0 and setting h(t) = 0 for t < 0. In addition h(t) is ser to zero for large values of t. The Sinc function is thus truncated at both ends. This truncation causes the frequency response of the filter to deviate from that of figure 16. Further details on how to design such (FIR) filters are in the EE4000 subject.